# DFT and FFT

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# The Outline



(Cont’d…)

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# State of the art

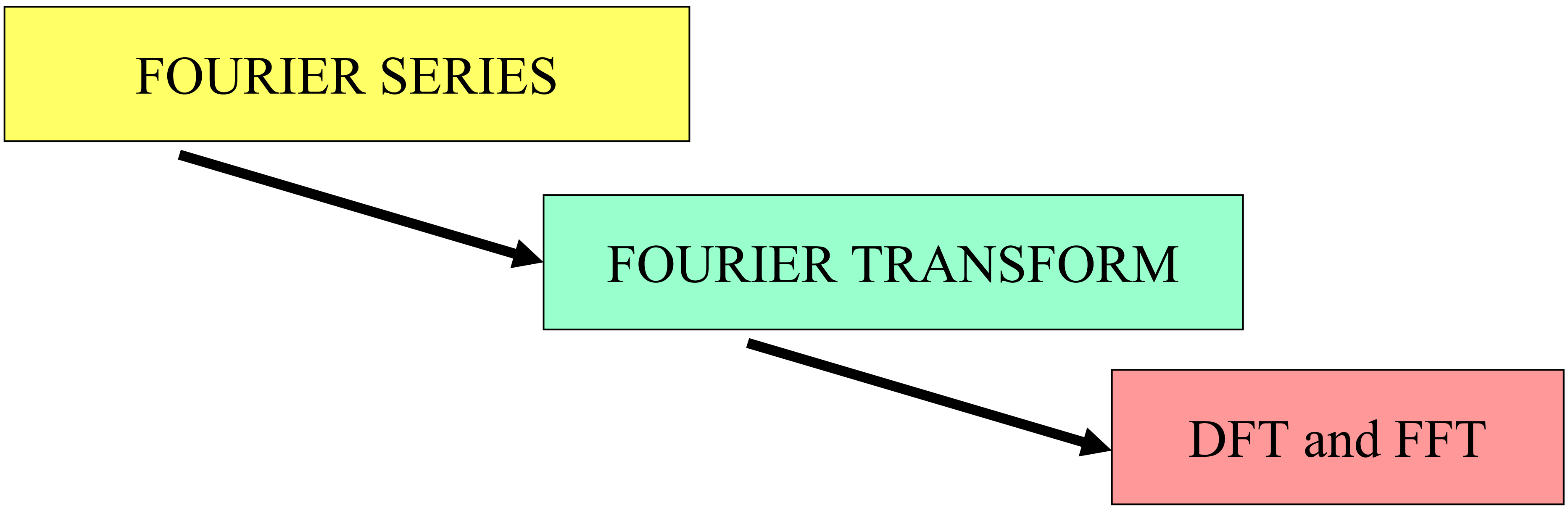
- The transformation of discrete data between the time and frequency is useful in analysis of the signal
- Fourier Transform is used for its transformation
- Voltage v.s. time become magnitude v.s. frequency and phase v.s frequency
- The domains provide complementary information about the same data

# The purpose of Discrete Transform

- Discrete transforms, particularly the discrete cosine transform are used in the data compression of speech and video signals to allow transmission with reduced bandwidth
- It is also used in image processing to obtain a reduced set of features for pattern recognition purposes
- For these computation the transformation from the frequency to the time domain is important

# What kind of transformation ?

- The Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT) are the best known and the most important



# Fourier Series

- Any periodic waveform , $f(t)$ can be represented as the sum of an infinite number of sinusoidal and cosinusoidal terms

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

- Remember that $f(t)$ is often a varying voltage v.s. time waveform

# Fourier series to the unit of volts

- Fourier series may be written more compactly by using exponential
- The series become relation of $d_n$ as the unit of volts, it is very useful in pulse graphic

$$f(t) = \sum_{n=-\infty}^{\infty} d_n e^{jn\omega t} \quad \Rightarrow \quad d_n = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} f(t) e^{-jn\omega t} dt$$

# The graphic example

- Here are the example, look at the periodic unipolar pulse waveform shown in figure 4.1 (a) . Deliberate choice of time origin to be offset form the centre and edge of a pulse is intended to allow illustration of the phase feature of the Fourier series .
- By substituting appropriate values into the formula of $d_n$, we can get the graphic below :

# The graphic example (cont'd)

Waveform

Amplitude Spectrum

Phase Spectrum

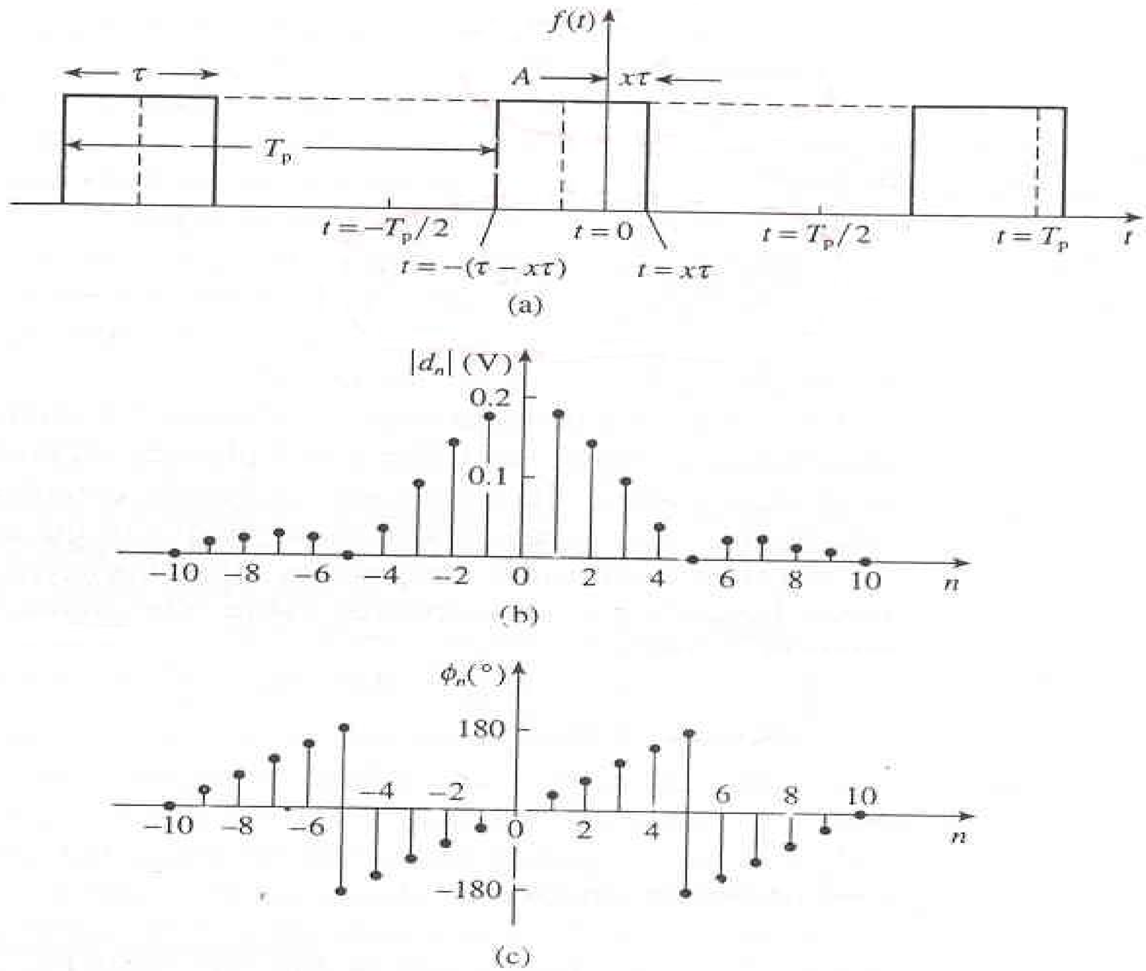



Figure 4.1

# The Fourier Transform

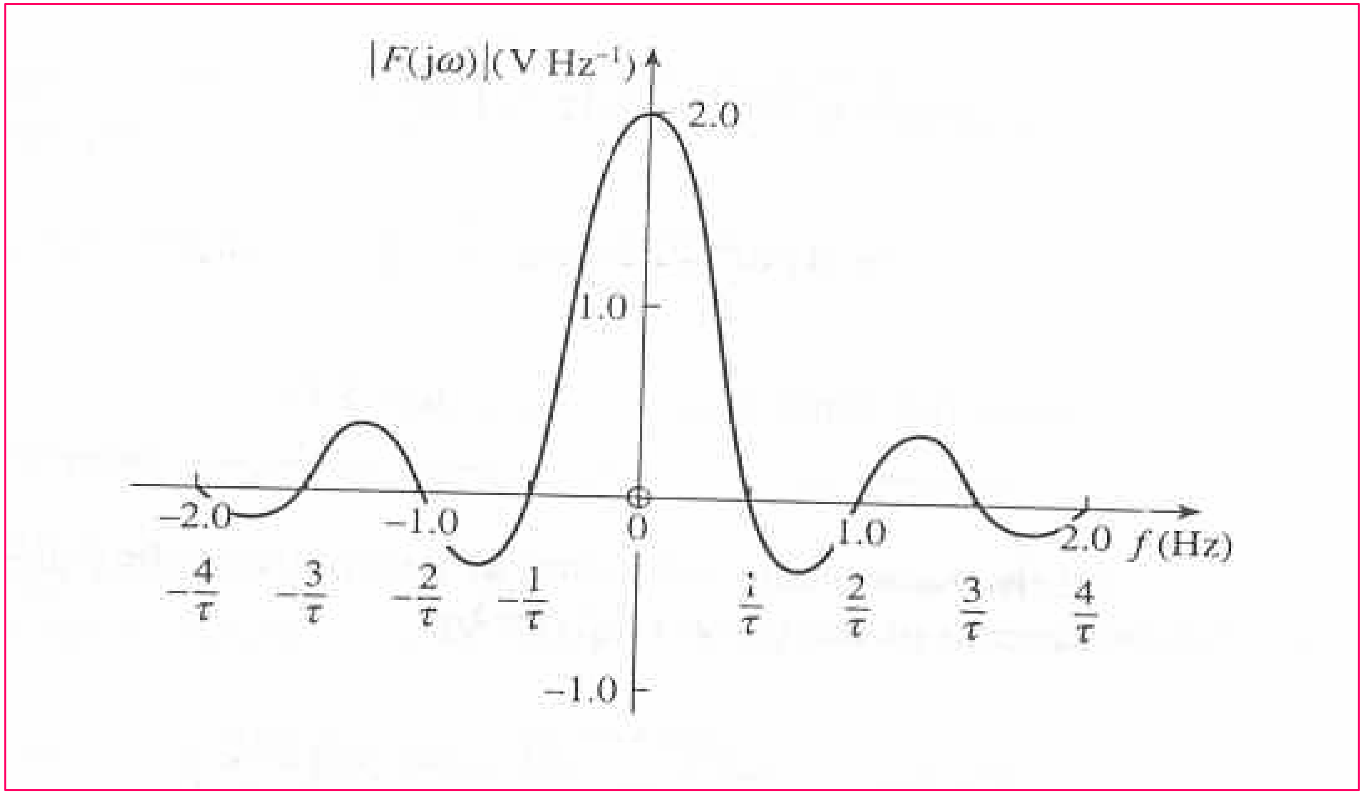
- The Fourier series approach has to be modified when the waveform is not periodic
- By using the following formula, we can change from the discrete frequency variable $n\omega$ to the continuous variable $\omega$
- So that, the amplitude and phase spectra become continuous

$$F(j\omega) = \frac{d(\omega)}{d\omega / 2\pi} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

# What is its transform ?

- *F*(*j* $\omega$) is complex and is known as the Fourier integral or more commonly = Fourier Transform
- Consider the graphic example on Figure 4.1 , by using Fourier transform we can transform *amplitude discrete pulse* to be *amplitude spectrum*
- Look at figure 4.2 on the next slide
- This is the key term of Fourier Transform that can be improved to be DFT and FFT

# Graphic : amplitude spectrum of a 2 V pulse




Amplitude Spectrum

Figure 4.2 a

# Graphic : energy spectrum of a 2 V pulse

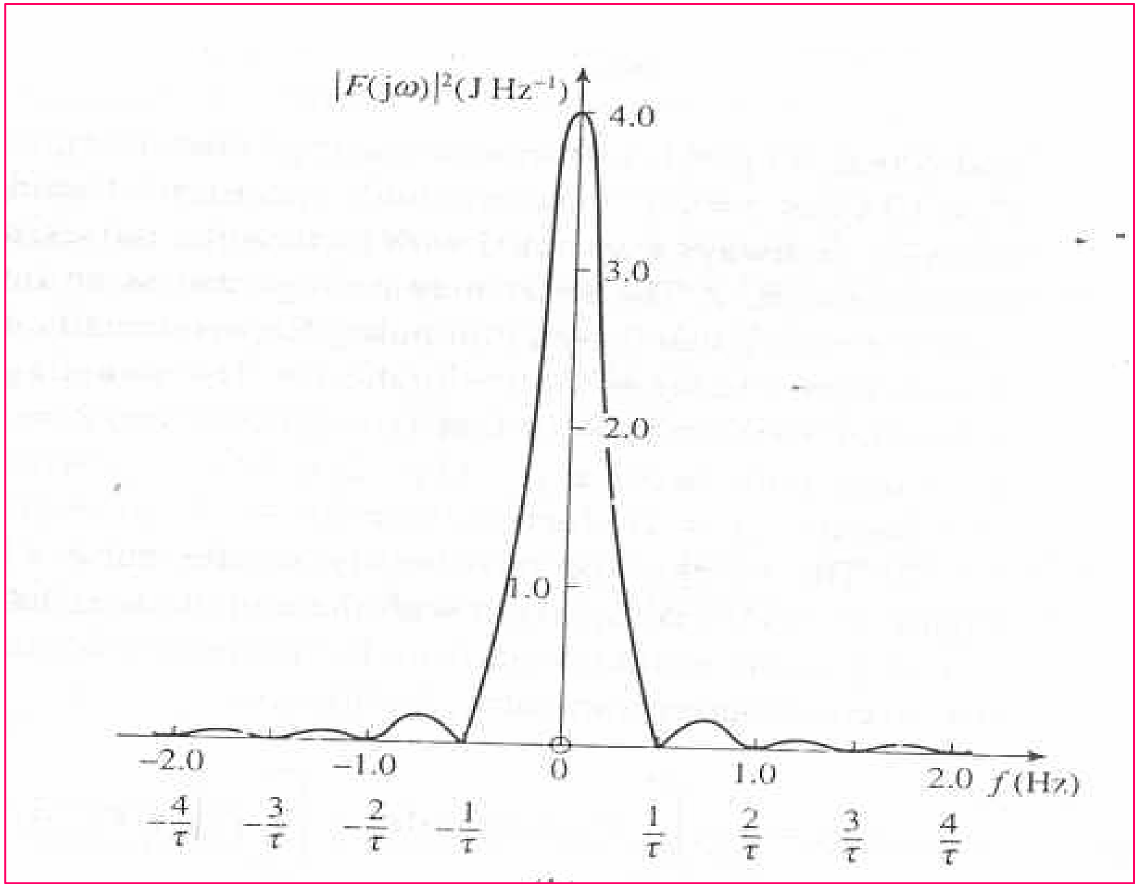



Figure 4.2 b

Energy Spectrum

# DFT and its inverse

- In practice the fourier components of data are obtained by digital computation rather than by analog processing
- This is achieved using a sample-and-hold circuit followed by an AD converter
- The problem is Fourier Transform can be used only for continuous data, while the data is commonly discrete and probably non-periodic

# DFT and its inverse (cont'd)

- However there is an analog transform for use with discrete data, known as Discrete Fourier Trans-form (DFT)
- The first assumption is : consider a waveform has been sampled at regular time intervals $T$ to produce the sample sequence :

$$\{x(nT)\} = x(0), x(T), \ldots , x[(N-1)T]$$

- Where $n$ is the sample number from $n=0$ to $n=N-1$

# DFT and its inverse (cont'd)

- The data values $x(nT)$ will be real only when representing the values of a time series such a voltage waveform
- The DFT of $x(nT)$ is then defined as the sequence of complex values

$$\{X(k\Omega)\} = X(0), X(\Omega), X(2\Omega), \ldots, X[(N-1)\Omega]$$

- Note that $\Omega$ is the first harmonic frequency given by $\Omega = 2\pi/NT$

# DFT and its inverse (cont'd)

- Related to our formula on previous slides, we can compare those equations, so that we have the DFT values $X(k)$ , is given by :

$$X(k) = F_D\left[x(nT)\right] = \sum_{n=0}^{N-1} x(nT)e^{-jk\Omega nT}$$

- Where $k$ = 0, 1, … , $N$ -1
- Watch out the exponent variable, don't miss it

$$-jk\Omega nT$$

# DFT and its inverse (cont'd)

- Because of $\Omega = 2\pi/NT$ , then the exponent variable of that formula can be convert to :

$$X(k) = \sum_{n=0}^{N-1} x(nT) e^{-jk\Omega nT}$$

$$X(k) = \sum_{n=0}^{N-1} x(nT) e^{-j\frac{2\pi nk}{N}}$$

# DFT and its inverse (cont'd)

- To make computation more easy, these equation are very useful :

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

- So that, DFT is the key for having transformation from time to frequency domain

# Example of DFT

Evaluate the DFT of sequence {1,0,0,1}

Answer :

Assume that this data represent four consecutive Voltages $x(0)=1$, $x(T)=0$, $x(2T)=0$, $x(3T)=0$

The data recorded at time intervals, $T$ and $N=4$.

Since $N$-1=3, then it is required to find the complex values of $X(k)$ for $k=0$, $k=1$, $k=2$, $k=3$,

Define each variable

$$X(k) = \sum_{n=0}^{N-1} x(nT) e^{-j\frac{2\pi n k}{N}}$$

# Example of DFT (cont'd)

For $k$ =0

$$X(0) = \sum_{n=0}^{3} x(nT)e^{-j\frac{2\pi.n.0}{N}}$$

$$= x(0) + x(T) + x(2T) + x(3T)$$

$$= 1 + 0 + 0 + 1$$

$$= \boxed{2}$$

Watch that $X(0)$=2 is entirely real, of magnitude 2
And phase angle $\Phi(0)$=0

# Example of DFT (cont'd)

For $k = 1$

$$X(1) = \sum_{n=0}^{3} x(nT) e^{-j\frac{2\pi . n . 1}{N}}$$

$$= 1.e^{-j\frac{2\pi . 0 . 1}{4}} + 0.e^{-j\frac{2\pi . 1 . 1}{4}} + 0.e^{-j\frac{2\pi . 2 . 1}{4}} + 1.e^{-j\frac{2\pi . 3 . 1}{4}}$$

$$= 1 \quad + 0 \quad + 0 \quad + 1.e^{-j\frac{2\pi . 3}{4}}$$

$$= 1 + \cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right)$$

$$= 1 + j$$

Magnitude = $\sqrt{2}$ Phase = $45^{o}$

# Example of DFT (cont'd)

For $k = 2$

$$X(1) = \sum_{n=0}^{3} x(nT) e^{-j\frac{2\pi . n . 2}{N}}$$

$$= 1.e^{-j\frac{2\pi .0.2}{4}} + 0.e^{-j\frac{2\pi .1.2}{4}} + 0.e^{-j\frac{2\pi .2.2}{4}} + 1.e^{-j\frac{2\pi .3.2}{4}}$$

$$= 1 + 0 + 0 + 1.e^{-j3\pi}$$

$$= 1 - 1$$

$$= 0$$

# Example of DFT (cont'd)

For $k=3$

$$X(1)=\sum_{n=0}^{3} x(nT)e^{-j\frac{2\pi.n.\,3}{N}}$$

$$=1.e^{-j\frac{2\pi.0.3}{4}}+0.e^{-j\frac{2\pi.1.3}{4}}+0.e^{-j\frac{2\pi.2.3}{4}}+1.e^{-j\frac{2\pi.3.3}{4}}$$

$$=1 \quad +0 \quad +0 \quad +1.e^{-j\frac{9\pi}{2}}$$

$$=1-j$$

Magnitude = √2 Phase = -45$^{o}$

# Example of DFT (cont'd)

From those computation we have the DFT for time series {1,0,0,1}, given by the complex sequence

$X(0) = 2$

$X(1) = 1 + j$

$X(2) = 0$

$X(3) = 1 - j$

$\text{DFT} = \{\, 2 \,,\, 1 + j \,,\, 0 \,,\, 1 - j \,\}$

PLOT THE GRAPHIC

# Cyclical property of DFT

- The fact that $X(k+N)=X(k)$
- It means DFT is periodic with period $N$
- The DFT components are repetitive
- This is the cyclical property of DFT
- The amplitude spectrum of an $N$-point DFT is symmetrical about harmonic $N/2$ _when both the zero and $(N+1)$th harmonics are included in the plot

So that.. what is N point DFT ?

# IDFT as its inverse

- Inverse DFT is very useful in transformation from the frequency to the time domain
- The value of IDFT is $x(nT)$ that commonly defined as $F_D^{-1}[X(k)]$

$$F_D^{-1}\left[X(k)\right] = \frac{1}{N}\sum_{k=0}^{N-1} X(k).e^{j\frac{2\pi.n}{N}k}$$

Please try to invert the previous example

# Properties of DFT

1. Symmetry

Re $[X(N\text{-}k)]$ =Re $X(\mathrm{k})$     where Re is real part

2. Even functions

If $x_e(n)$ is an even function, then : $x_e(n) = x_e(-n)$

$$F_D\left[x_e(n)\right] = \sum_{n=0}^{N-1} x_e(n)\cos\left(k\Omega nT\right)$$

# Properties of DFT

## 3. Odd functions

If $x_o(n)$ is an even function, then : $x_o(n) = -x_o(-n)$

$$F_D\left[x_o(n)\right] = -j.\sum_{n=0}^{N-1} x_o(n)\sin\left(k\Omega nT\right)$$

## 4. Parseval's Theorem

$$\sum_{n=0}^{N-1} x^2(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left|X(k)\right|^2$$

← The normalized energy In the signal

# Properties of DFT

## 5. Delta Function

$$F_D\left[\delta(nT)\right] = 1$$

## 6. The linear cross-correlation computed using DFT

$$r_{c\,x1\,x2}(j) = F_D^{-1}[\,X_1^*(k)\,X_2(k)\,]$$ → Circular correlation

$$r_{x1\,x2}(j) = F_D^{-1}[\,X_{1a}^*(k)\,X_{2a}(k)\,]$$ → Linear-cross corr.

# Computational Complexity of DFT

- A large number of multiplications and additions are required for the calculations of DFT
- For $N$-point DFT there will be $N^2$ multiplications and $N.(N-1)$ additions
- Just imagine for thousands of data there will be millions of multiplications and additions, that needs a lot of memory, time and cost
- So what should we do for reducing this number ?

# Fast Fourier Transform (FFT)

- FFT is an algorithm that is useful for speeding up the computation
- When applied in the time domain, the algorithm is referred to as a decimation-in-time (DIT) FFT
- Decimation refers to the significant reduction in the number of calculations performed on time domain data
- The computational savings will be seen to increase as $N^2 - (N / 2) \log_2 N$

# FFT Algorithm

- The notation can be re-written as :

$$X_1(k) = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi \ n \ k}{N}} \qquad k = 0,\ldots, N-1$$

- The factor $e^{-j\,2\pi\,/\,N}$ is written as $W_{N,\,,}$, so that :

$$X_1(k) = \sum_{n=0}^{N-1} x_n W_N^{k \ n} \qquad , k = 0,\ldots, N-1$$

# FFT Algorithm (cont'd)

- Remember these useful property :

$$W_N = e^{-j2\pi/N}$$

$$W_N^2 = W_{N/2}$$

$$W_N^{(k+N/2)} = -W_N^k$$

Find out where do these come from

- The data sequences $X_1$ is divided into two equal sequences (even and odd ) $X_{11}$ and $X_{12}$

# FFT Algorithm (cont'd)

- Because of those property, we can say that the DFT $X_1(k)$ can be expressed in terms of two DFTs : $X_{11}(k)$ and $X_{12}(k)$ with the factor $W_{N/2}{}^{k}$

- In the equation $\boldsymbol{X_1(k) = X_{11}(k) + W_{N/2}{}^{k} \cdot X_{12}(k)}$

- The number of $k$ is only from 0 to $N$-1
- The factor $W_N^k$ needs calculation once only

# The Butterfly Method

- The FFT algorithm is applied in Butterfly method
- This method gives a very simple calculation to determine the value of DFT
- You can compare the time that you need for having that value between '*DFT classic method* ' and '*FFT Butterflies method*'
- The point is you have to know exactly the FFT butterflies, at least for 8-point DFT on the figure below :

# The FFT Butterflies for an 8-point DFT

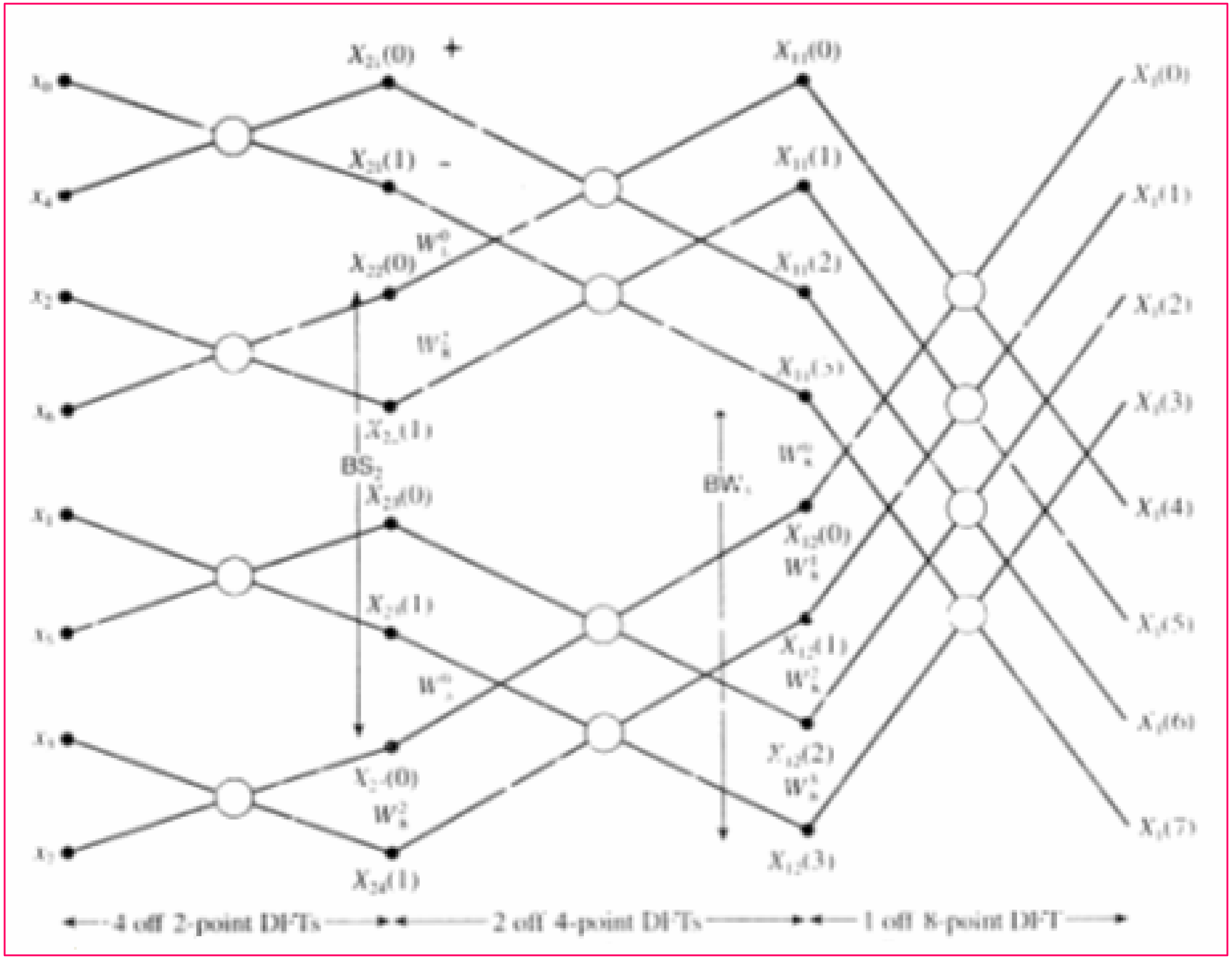



Figure 4.3

# Example for 4-point DFT

Find out the DFT of a sequence {1, 0, 0, 1}

Answer

Note that its only 4-point DFT, so that from the previous figure points $x_0$, $x_4$, $x_2$, $x_6$ are replaced by $x_0$, $x_2$, $x_1$, $x_3$ and the require DFT values are :

$X_{11}(0)$, $X_{11}(1)$ , $X_{11}(2)$, $X_{11}(3)$,

Therefore our calculation only up to second step

# Example for 4-point DFT (cont'd)

$$X_{21}(0) = x_0 + x_2 = 1$$
$$X_{21}(1) = x_0 - x_2 = 1$$
$$X_{22}(0) = x_1 + x_3 = 1$$
$$X_{22}(1) = x_1 - x_3 = -1$$

$$X_{11}(0) = X_{21}(0) + W_8^0 X_{22}(0) = 1 + 1 = 2$$
$$X_{11}(1) = X_{21}(1) + W_8^2 X_{22}(1) = 1 + e^{-j\pi/2}(-1) = 1 + j$$
$$X_{11}(2) = X_{21}(0) - W_8^0 X_{22}(0) = 1 - 1 = 0$$
$$X_{11}(3) = X_{21}(1) - W_8^2 X_{22}(1) = 1 - j$$

# Example for 4-point DFT (cont'd)

From those value we know that those are the same as the value that we determine from the 'DFT classical method'
So that, we may say : By using FFT, the computational savings, increase as the number of data increases

What about if you try by using computer program such as C++ or MATLAB ?

# Applications of the DFT and FFT

DO YOU KNOW WINAMP ?

→ YES OF COURSE, I ALWAYS LISTEN BEFORE I GO TO BED

DO YOU KNOW IT IS ONE OF THE **DFT** IMPLEMENTATION

→ NO , I DON'T

FIND OUT HOW ITS RELATION

→ ? ? ? ? ? ? ? ?

# End of this session

PREPARE
FOR
THE REVIEW

Before that…
ARE YOU INTEREST WITH THIS TOPIC ?

# Review

1. Do the exercise on text book [ *Ifeachor* ] page 158-160. The number that you've to do is related to your absence number, <u>but now its reversed</u>
2. Repeat example 3.3 on text book [*Ifeachor*] page 114, by using MATLAB. Print out the result.

You have to be able for using other program,
not only MATLAB